# Petri nets with generalized algebra: a comparison

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**Abstract**

In the last decade we can see substantial effort to develop an abstract and uniform constructions for Petri nets. Most of such abstractions are based on algebraic characterizations of Petri nets. They work mostly over commutative monoids and their various subclasses, namely cancellative commutative monoids or cones of Abelian groups. In the paper we study relationships between Petri nets with generalized underlying algebra. More precisely, we study Petri nets over commutative monoids, cancellative commutative monoids, cones of Abelian groups, and fully ordered cones of Abelian groups. As the main result, we show that classes of reachability graphs of Petri nets over cancellative commutative monoids and cones of Abelian groups coincide (up to isomorphism). In other words, partial order on used cancellative commutative monoid plays no role in expressive power of Petri nets. However, as shows the fact that the class of reachability graphs of nets over fully ordered cones is a proper subclass of the class of reachability graphs of nets over cancellative commutative monoids, the total order on used monoids plays an important role in expressive power of Petri nets.

## 1 Introduction

During the last three decades Petri nets became an accepted platform for modelling and analysis of concurrent systems.

A lot of effort has been put into extensions of Petri nets. Some of such extensions were developed in order to enable more effective modelling without

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extension of the modelling power. These extensions are also referred as compressions of Petri nets. A typical example of such compressions are different kinds of high-level Petri nets [9].

Other type is represented by extensions developed with the purpose of increasing the modelling power - in this paper also referred as behavioral extensions of Petri nets. There exist at least three principal approaches of behavioral extensions. The first one is characterized by modification of the enabling rule, and may be typically represented by Petri nets with inhibitor arcs [20,8], probably the most known behavioral extension, but also recently developed Contextual nets [17], or Controlled Petri nets used in discrete event control [7]. The second approach of behavioral extensions is based on functions as weights of arcs connecting places and transitions of the net. Here belong self-modifying nets [22] and their further generalizations. The third approach may be characterized by generalization of algebra used in Petri nets. This direction was open in the seminal paper [16], and is becoming still more popular. Moreover, there are already attempts to combine extensions based on modification of enabling rule and generalization of the net underlying algebra [11–13]. Certainly, there may be found also other directions of Petri net extensions than the mentioned compressions and behavioral extensions, such as extensions introducing time.

In this paper we study behavioral extensions characterized by generalization of the underlying algebra. Let us notice, that place/transition nets (shortly p/t nets), a basic version of Petri nets, use free commutative monoids over the set of places[2] as the underlying algebra. In [16] the possibility of using an arbitrary commutative monoid was pointed out. Later, in [19] commutative semigroups were considered. Cancellative commutative monoids are suggested in [10], while cones of groups are used in series of papers [3,2,21]. In particular, [5] uses cones of Abelian groups. The reasons for the study of generalizing the Petri net underlying algebra lies in practice and in theory as well. Namely, engineers use different algebra than free commutative monoids for more than ten years, for instance reals with addition in Continuous Petri nets [4] and Hybrid Petri nets [1], or different kinds of logic in fuzzy and logic Petri nets [14]. Using different algebra one may remove disadvantages of standard p/t nets which cannot model cyclic behavior with just one transition [11].

The properties of free commutative monoids used in standard p/t nets cause that reachability graphs of standard p/t nets have some specified properties. Now it seems to be natural to investigate which of these properties are preserved in p/t nets with specifically generalized algebra. Thus, the plan of the paper is to set a tuple of properties, namely determinism and commutativity of the transitions, absence of simple cycles (of length greater than one), and a technical property called the comb, which hold in reachability

[2] *i.e.*, multi-sets of places with multi-set addition, or in other words nonnegative integer vectors with addition

graphs of all standard p/t nets, and then to investigate which of them are preserved in classes of p/t nets over a specified underlying algebra. Based on this investigation we state the containment relation for these classes.

Our main result in this line is a bit countering to our earlier intuition: We show that the class of reachability graphs of Petri nets over cancellative commutative monoids coincides with the class of reachability graphs of Petri nets over cones of Abelian groups. However, by considering fully ordered cones of Abelian groups we get a proper subclass of the class of reachability graphs of Petri nets over cancellative commutative monoids. So, while cones of Abelian groups do not express a border where some properties of reachability graphs may be lost, fully ordered cones express such a border. In other words, partial order on used cancellative commutative monoid plays no role in expressive power of Petri nets. However, as shows the fact that the class of reachability graphs of nets over fully ordered cones is a proper subclass of the class of reachability graphs of nets over cancellative commutative monoids, the total order on used monoids plays an important role in expressive power of Petri nets. Another simple, but a bit surprising results we illustrate in a simple example is that reachability graphs of Petri nets over non-cancellative commutative monoids may be nondeterministic.

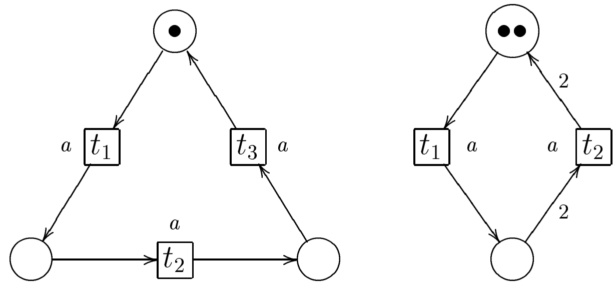
Now, one may ask why we have selected the investigated properties. In the following we try to give a motivation for choosing these properties as well as to draft the significance of the obtained results.

Petri nets are popular, among others, because of the linear algebraic analytic techniques they offer. This techniques are based on the fact that in standard p/t nets single transitions cause deterministic changes of the markings, and moreover, one may use multi-sets of transitions instead of sequences of transitions in a deterministic way. Thus, the commutativity is just the property which permits to overcome sequentiality of computations in standard p/t nets! So, determinism and commutativity are properties crucial for using popular linear algebraic techniques such as the state equation and invariants of Petri nets. Now, showing that this properties may be lost in nets over non-cancellative commutative monoids (such as power-set with union) we say that overstepping the border given by cancellativity of the used algebra we lose popular and powerful linear algebraic techniques.

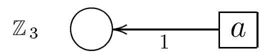
On the other hand, absence of simple cycles in reachability graphs of p/t nets can be considered as a serious complication. For instance, let us take the transition system $\mathcal{S}$ with three states $\{s_1, s_2, s_3\}$ and just one label, *i.e.*,, just one atomic action $t$, given as follows:

$$\begin{tikzcd}
 & s_1 \arrow[dl, "t"'] & \\
s_2 \arrow[rr, "t"] & & s_3 \arrow[ul, "t"']
\end{tikzcd}$$

Clearly, there does not exist any p/t net whose sequential case graph is (isomorphic to) the given transition system $\mathcal{S}$, because for an arbitrary set of places $P$ all nonzero elements of the commutative monoid of multi-sets over $P$ with multi-set addition have infinite order. In other words, the system $\mathcal{S}$ cannot be modeled by any p/t net with just one transition. To model this system by a Petri net, it is obvious to use one of the following labeled Petri nets[3], with three, or at least two transitions labeled by the same label.




However, using a cyclic group of order 3 instead of free commutative monoid, such system would be modeled by the following unlabeled Petri net with just one transition.




Considering more complex systems with cyclic behavior, *e.g.* transition systems with many cycles of different order, where each cycle is labeled using just one label, the number of necessary transitions in a labeled Petri net model may combinatorially grow with respect to the number of labels in the modeled system. To avoid this undesirable growth, it would suffice to use cyclic groups of the related orders. So, the presented results show that using cancellative commutative monoids we can remove this disadvantage (*i.e.* avoid this undesirable growth) and still preserve such nice properties as determinism and commutativity, which are crucial for linear algebraic techniques.

The selection of the comb property has a more general background. Determinism, commutativity and absence of simple cycles are properties with the following characteristics: when they hold in a reachability graph, then they hold in every its subgraph. Evidently, properties with this characteristics cannot be lost by any change of the enabling rule in p/t nets (*e.g.* cannot be lost by allowing inhibitor arcs). Therefore such properties can be said of purely algebraic nature. The comb property serves as an example of the different class of properties which can be lost in subgraphs of reachability graphs of standard p/t nets, *i.e.*, they can be lost by a change of the enabling rule. However, as the comb property illustrates, some of these properties may be lost also by generalization of the underlying algebra. In other words, by generalization of the Petri net underlying algebra one can increase modeling power of Petri nets in several directions (one can relax properties of purely algebraic nature but also some of those removable by a change of the enabling rule).

[3] *i.e.*, a Petri net whose transitions are labeled by elements of a set of labels [20]

## 2 Preliminaries

Before defining of Petri nets, let us first define some notation. We use $\mathbb{Z}$ to denote integers, $\mathbb{Z}^+$ to denote positive integers, and $\mathbb{N}$ to denote nonnegative integers. Moreover, we also shortly write $\mathbb{Z}$ to denote the infinite cyclic group of integers with addition $(\mathbb{Z}, +)$, and $\mathbb{N}$ to denote the commutative monoid of nonnegative integers with addition $(\mathbb{N}, +)$. Given two arbitrary sets, say $A$ and $B$, symbol $B^A$ denotes the set of all functions from $A$ to $B$. Given a function $f$ from $A$ to $B$ and a subset $C$ of the set $A$ we write $f|_C$ to denote the restriction of the function $f$ on the set $C$. As usual, symbol $2^A$ denotes the power set (*i.e.* the set of all subsets) of the set $A$. To denote the set of all multi-sets over a set $A$ with multi-set addition we write $(\mathbb{N}^A, +)$ or shortly $\mathbb{N}^A$. To denote the set of all finite multi-sets over a set $A$ with multi-set addition, *i.e.*, the free commutative monoid over a set $A$, we write $(\mathbb{N}^A_{fin}, +|_{\mathbb{N}^A_{fin} \times \mathbb{N}^A_{fin}})$. Thus, $\mathbb{N}^A_{fin} = \{b \mid b \in \mathbb{N}^A \ \wedge \ |A_b| \in \mathbb{N}\}$, where $A_b = \{a \mid a \in A \ \wedge \ b(a) \neq 0\}$ is a subset of the set $A$ mapped by function $b \in \mathbb{N}^A$ to nonzero integers, *i.e.*, $A_b$ is the set of elements from $A$ that are contained (occur at least once) in the multi-set $b$. Clearly, $(\mathbb{N}^A_{fin}, +|_{\mathbb{N}^A_{fin} \times \mathbb{N}^A_{fin}})$ is a sub-monoid of the monoid $(\mathbb{N}^A, +)$. Similarly, we use $(\mathbb{Z}^A, +)$ or shortly $\mathbb{Z}^A$ to denote the Abelian group of integer vectors over a set $A$ with element by element addition. To denote the free Abelian group over a set $A$ we write $(\mathbb{Z}^A_{fin}, +|_{\mathbb{Z}^A_{fin} \times \mathbb{Z}^A_{fin}})$. Thus, $\mathbb{Z}^A_{fin} = \{b \mid b \in \mathbb{Z}^A \ \wedge \ |A_b| \in \mathbb{N}\}$, where $A_b$ is given as previously. Evidently, the free Abelian group $(\mathbb{Z}^A_{fin}, +|_{\mathbb{Z}^A_{fin} \times \mathbb{Z}^A_{fin}})$ is a subgroup of Abelian group $(\mathbb{Z}^A, +)$. As usual, we write only $(\mathbb{N}^A_{fin}, +)$ or shortly $\mathbb{N}^A_{fin}$ instead of rather complicated $(\mathbb{N}^A_{fin}, +|_{\mathbb{N}^A_{fin} \times \mathbb{N}^A_{fin}})$; and $(\mathbb{Z}^A_{fin}, +)$ or shortly $\mathbb{Z}^A_{fin}$ instead of $(\mathbb{Z}^A_{fin}, +|_{\mathbb{Z}^A_{fin} \times \mathbb{Z}^A_{fin}})$. Notice that if the set $A$ is finite, then $\mathbb{N}^A = \mathbb{N}^A_{fin}$ and $\mathbb{Z}^A = \mathbb{Z}^A_{fin}$. As one may see from the previous notation, we often use the symbol $+$ in the paper universally to denote a binary operation, *i.e.*, we use the symbol $+$ for different operations. Finally, we use symbol $\oplus$ to denote direct product of monoids or groups.

**Definition 2.1** A *place/transition net* (shortly a *p/t net*) is an ordered tuple $\mathcal{N} = (P, T, I, O)$, where $P$ and $T$ are non-empty distinct sets of *places* and *transitions*; $I : T \to \mathbb{N}^P$ is an *input function*; $O : T \to \mathbb{N}^P$ is an *output function*. A *marked place/transition net* (shortly a *marked p/t net*) is an ordered tuple $\mathcal{MN} = (\mathcal{N}, M_0)$, where $\mathcal{N}$ is a p/t net; and $M_0 \in \mathbb{N}^P$ is an *initial marking*.

A *state* of a p/t net $\mathcal{N}$ called *marking* and denoted by $M$ is a multi-set over $P$, *i.e.*, an element of the monoid $\mathbb{N}^P$. The dynamics of the net is expressed by *occurrence* (*firing*) of *enabled* transitions. A transition $t \in T$ is enabled to occur in a marking $M \in \mathbb{N}^P$ iff $\forall p \in P : M(p) \geq I(t)(p)$ and then its occurrence leads to the marking

$$M' = M + O(t) - I(t). \tag{1}$$

Looking for the algebraic substance of Petri nets, we can easily reformulate this definition as follows. A p/t net is a quadruple $(\mathbb{N}^P, T, I, O)$, where $\mathbb{N}^P$ is the underlying algebra, and $T, I, O$ has the same meaning as in Definition 2.1[4]. Thus, a state of a p/t net is an element of the underlying algebra $\mathbb{N}^P$. A transition $t \in T$ is enabled to occur in a state $M \in \mathbb{N}^P$ iff $\exists X \in \mathbb{N}^P : X + I(t) = M$; and then its occurrence leads to the new marking $M'$ such that $M' = X + O(t)$.

So, if we agree that generalization of the underlying algebra of p/t nets is meaningful, we get the following simple and generic definition of p/t nets.

**Definition 2.2** A *place/transition net* (shortly a *p/t net*) *with generalized algebra* is an ordered tuple $\mathcal{GN} = (\mathcal{H}, T, I, O)$, where $\mathcal{H} = (H, +)$ is a groupoid, $T$ is a non-empty set of *transitions*; $I : T \to H$ is an *input function*; $O : T \to H$ is an *output function*. A *marked place/transition net* (shortly a *marked p/t net*) *with generalized algebra* is an ordered tuple $\mathcal{MGN} = (\mathcal{GN}, M_0)$, where $\mathcal{GN}$ is a p/t net with generalized algebra; and $M_0 \in H$ is an *initial marking*.

**Definition 2.3** A *state* of a p/t net with generalized algebra (also called *marking*) is an element of the carrier of the underlying algebra $\mathcal{H}$. A transition $t \in T$ is *enabled* to *occur* in a state $M \in H$ iff $\exists X \in H : X + I(t) = M$; and then its occurrence leads to the new marking $M'$ such that $M' = X + O(t)$.

In the following, a p/t net with generalized algebra is simply said to be a p/t net. In particular, a p/t net according to Definition 2.1, is said to be a standard p/t net.

In this paper, we will consider for the underlying algebra only associative and commutative groupoids with a neutral element, that means commutative monoids, as supposed in [16]. Here we have to mention that there exist also successfully used and already studied kinds of Petri nets which use non-commutative algebra, *e.g.* FIFO nets which use free monoids [21].

Despite the fact that we are focused just on the algebraic characterization of nets and we do not consider the problem of distribution, which is connected with the concept of places, in examples used in the paper we use similar graphical expression of marked p/t nets with generalized algebra as in the case of standard marked p/t nets. We draw a marked p/t net in the form of bipartite oriented graph, in which a circle (just one) contains written value of the initial state, transitions are drawn as boxes, and if $I(t) \neq 0$ ($O(t) \neq 0$) then we draw an arc from the circle to the box representing transition $t$ (from the box representing transition $t$ to the circle) labeled (weighted) by the value $I(t)$ ($O(t)$).

[4] In other words, as pointed in [16], p/t nets can be understood as graphs with arcs formed by set $T$ and sources and targets of these arcs given by input and output functions. Thus, vertices of such a graph are elements of the carrier of the net underlying algebra, which is in the case of standard p/t nets monoid of multi-sets over the set of places with multi-set addition.

**Remark 2.4** Here let us mention that one can deal with concept of places in nets over more general algebra than free commutative monoids in at least two different ways. In the first one, places represent subsystems of modeled system. Each place works with its own commutative monoid. Elements of this monoid represent local states (local markings) of the place representing the related subsystem. Elements of the direct product of monoids associated with places form the global states (global markings) of the net. Such approach may be seen to be application oriented, and from mathematical point of view such concept of places may be seen to be high-level in similar way with role of places in high-level Petri nets. Another natural concept is to consider generators of the underlying commutative monoid to be places of the net. From this point of view, generalization of algebra of Petri nets corresponds to addition of equations to the definition of nets. For simplicity, in presented examples we implicitly use the first concept considering just one subsystem.

At this point we recall the definition of labeled transition systems [23]:

**Definition 2.5** A *labeled transition system* (shortly *transition system*) is an ordered tuple $\mathcal{S} = (S, L, \longrightarrow)$, where $S$ is a set of *states*, $L$ is a set of *labels* and $\longrightarrow \subseteq S \times L \times S$ is a *transition relation.*

The fact that $(s, a, s') \in \longrightarrow$ is written as $s \xrightarrow{a} s'$. Denoting by $L^*$ the monoid of finite strings of labels from $L$ with concatenation, it is obvious to extend the transition relation to *string transition relation* $\longrightarrow_\star \subseteq S \times L^* \times S$ as follows: $(s, q, s') \in \longrightarrow_\star$ whenever there exists a, possibly empty, string of labels $q = a_1 ... a_n$ such that $s \xrightarrow{a_1} s_1 \cdots s_{n-1} \xrightarrow{a_n} s'$. To denote $(s, q, s') \in \longrightarrow_\star$ we simply write $s \xrightarrow{q}_\star s'$. A state $s'$ is said to be *reachable* from a state $s$, iff there exists a string of labels $q$ such that $s \xrightarrow{q}_\star s'$. Given a state $s \in S$, the set of all states reachable from $s$ is denoted by $\{s \longrightarrow_\star\}$.

**Definition 2.6** A *pointed labeled transition system* (shortly *pointed transition system*) is an ordered tuple $\mathcal{PS} = (\mathcal{S}, s_0)$, where $\mathcal{S} = (S, L, \longrightarrow)$ is a labeled transition system; and $s_0 \in S$ is a distinguished *initial state* such that $\{s_0 \longrightarrow_\star\} = S$, *i.e.*, every state is reachable from initial state $s_0$.

Now it is straightforward to see that each p/t net can be associated with a labeled transition system.

**Definition 2.7** Let $\mathcal{GN} = (\mathcal{H}, T, I, O)$ be a p/t net. Then the labeled transition system $\mathcal{S} = (H, T, \longrightarrow)$ such that $M \xrightarrow{t} M' \iff \Big(t$ is enabled to occur in $M$ and its occurrence leads to the marking $M'\Big)$ is called *reachability graph* of the p/t net $\mathcal{GN}$.

Given any marking $M_0 \in H$, the pointed labeled transition system $\mathcal{PS} = (\{M_0 \longrightarrow_\star\}, T, \longrightarrow \cap \{M_0 \longrightarrow_\star\} \times T \times \{M_0 \longrightarrow_\star\}, M_0)$ is called *reachability graph* of the marked p/t net $\mathcal{MGN} = (\mathcal{GN}, M_0)$.

In the following, we choose a tuple of properties, that hold in the reachability graph of each standard marked p/t net given by Definition 2.1.

Recall that given a finite sequence $q = a_1 \dots a_n$ over a set $A$ we write $b_q$ to denote Parikh's image of $q$, *i.e.*, $b_q \in \mathbb{N}^A_{fin}$ is a multi-set in which the number of the occurrences $b_q(a)$ of each element $a$ from the set $A$ is given by the number of its occurrences in $q$, formally $b_q(a) = |\{i \mid i \in \{1, \dots, n\} \wedge a_i = a\}|$ for every $a \in A$.

**Definition 2.8** Let $\mathcal{P}S = (S, L, \longrightarrow, s_0)$ be a pointed labeled transition system. We say that $\mathcal{P}S$ is called *deterministic* iff $\forall s \xrightarrow{a} s', s \xrightarrow{a} s'' : s' = s''$. We say that $\mathcal{P}S$ is *commutative* iff $\forall\ s \xrightarrow{q}_\star s', s \xrightarrow{q'}_\star s'' : b_q = b_{q'} \Rightarrow s' = s''$. We say that $\mathcal{P}S$ preserves the *comb property* iff for all states $s, s', s_1, s_1', s_2, s_2'$ of $\mathcal{PS}$ and every pair of labels $a_1, a_2 \in L$ the following condition holds:

$$\begin{tikzcd} s \arrow[r, "a_2"] \arrow[d, "a_1"'] & s' \\ s_1 \arrow[d, "a_1"'] & \\ s_2 \arrow[r, "a_2"] & s_2' \end{tikzcd} \qquad \Longrightarrow \qquad \exists s_1' \in S : s_1 \xrightarrow{a_2} s_1'$$

We say that $\mathcal{P}S$ *contains no simple cycles* iff for all elementary cycles $s \xrightarrow{q}_\star s$ (*i.e.* such cycles that there does not exist any nonempty proper prefix $q'$ of string $q$ such that $s \xrightarrow{q'}_\star s$) there holds:

$$(\exists a \in L : b_q(a) > 1) \Longrightarrow (\exists a' \in L : a \neq a' \wedge b_q(a') > 0)$$

**Corollary 2.9** *Every commutative labeled transition system is deterministic.*

**Lemma 2.10** *The reachability graph of every standard marked p/t net given by Definition 2.1 is deterministic, commutative, preserves the comb property and contains no simple cycles.*

**Proof.** *Determinism:* it goes directly from Equation 1;
*Commutativity:* Given a function $f : T \to \mathbb{Z}^P$, we denote by $\hat{f}$ the linear $\mathbb{N}$-extension of the function $f$, *i.e.*, we have $\hat{f} : \mathbb{N}^T_{fin} \to \mathbb{Z}^P$ is such that $\forall b \in \mathbb{N}^T_{fin} : \hat{f}(b) = \sum_{t \in T_b} f(t) \cdot b(t)$ [5], where naturally the sum of empty set is zero-function, *i.e.*, we define $\hat{f}(0) = 0$. Properties of the free Abelian group $\mathbb{Z}^P$ over a set $P$ enable us to write $M' = M + \hat{O}(b_q) - \hat{I}(b_q)$ whenever $M \xrightarrow{q}_\star M'$. Now, take any $M \xrightarrow{q}_\star M', M \xrightarrow{q'}_\star M''$ such that $b_q = b_{q'}$. From the previous result we have directly that $M' = M + \hat{O}(b_q) - \hat{I}(b_q) = M + \hat{O}(b_{q'}) - \hat{I}(b_{q'}) = M''$.
*The comb property:* Assume a situation that comb property does not hold, *i.e.*,

$$\forall p \in P : M(p) \geq I(t_2)(p) \wedge \tag{2}$$
$$\exists p \in P : M(p) + O(t_1)(p) - I(t_1)(p) < I(t_2)(p) \wedge \tag{3}$$
$$\forall p \in P : M(p) + O(t_1)(p) \cdot 2 - I(t_1)(p) \cdot 2 \geq I(t_2)(p) \tag{4}$$

[5] where $T_b$ is the support of multi-set $b$, *i.e.*, $T_b = \{t \mid t \in T \wedge b(t) \neq 0\}$

But then for such $p$ that (3) holds we have in the case $O(t_1)(p) - I(t_1)(p) \leq 0$ that (3) would be in contradiction with (4). Otherwise, in the case $O(t_1)(p) - I(t_1)(p) > 0$, inequality (3) would be in contradiction with (2).
*Absence of simple cycles:* it goes directly from the torsion-freeness[6] of the free Abelian group $\mathbb{Z}^P$. $\square$

## 3 Relationship between Petri nets over commutative monoids

One of the first works which discusses p/t nets with generalized underlying algebra is the seminal paper [16], where in general case commutative monoids are considered. As the following example shows, the commutative monoids when used as a p/t net underlying algebra may cause that the related reachability graph becomes non-deterministic, and hence according to the corollary non-commutative.

**Example 3.1** Given a set $P = \{a, b\}$, take for the domain of markings non-cancellative commutative monoid formed by the power set of $P$ with standard union, *i.e.*, $(2^P, \cup)$. Let $T = \{t\}, I(t) = \{a, b\}$ and $O(t) = \emptyset$. Then we have that occurrence of $t$ in initial marking $M = \{a, b\}$ leads to the marking $M'$ such that $M' \cup \{a, b\} = M$. From the following figure with the reachability graph of such net we can see that occurrence of $t$ is non-deterministic, because in the marking $\{a, b\}$ it may lead again to $\{a, b\}$, but also to markings $\emptyset$, $\{a\}$ or $\{b\}$.

$$\begin{tikzcd}
 & \{a,b\} \arrow[loop above, "t"] \arrow[dl, "t"'] \arrow[d, "t"'] \arrow[dr, "t"] & \\
\emptyset & \{a\} & \{b\}
\end{tikzcd}$$

The loss of determinism and therefore also commutativity is caused by non-uniqueness of the $\cup$-complement in the used monoid. However, if one uses cancellative commutative monoid[7], as it is suggested in [10], the determinism as well as commutativity of the reachability graphs are preserved.

**Lemma 3.2** *Reachability graph of every p/t net over cancellative commutative monoid is deterministic and commutative.*

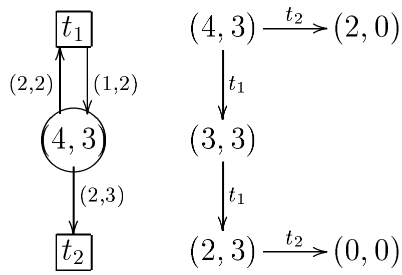
**Proof.** Recall that a cancellative commutative monoid $(H, \dot{+})$ may be embedded into an Abelian group, *i.e.*, there exists an Abelian group $(G, +)$ such that $H \subseteq G$ and the restriction of the group operation $+$ on set $H$ is equal

[6] Recall that a group $(G, +)$ is said to be *torsion free* iff every nonzero element in $G$ has infinite order, *i.e.*, iff $\forall g \in G : g \neq 0 \Longrightarrow (\forall k \in \mathbb{Z}^+ : g \cdot k \neq 0)$
[7] Recall that a commutative monoid $(H, +)$ is *cancellative* iff $\forall a, b, c \in H : a + c = a + b \Longrightarrow b = c$

to the monoid operation $\dot{+}$. Using this group we can show determinism and commutativity in the same way as it is done in the proof of Lemma 2.10. $\square$

Now one may ask the question whether for all commutative monoids, and if not for all, whether for cancellative commutative monoids the comb property is preserved. As the following example illustrates the answer is negative.

**Example 3.3** In the following figure there is a marked p/t net over monoid $\mathcal{H} = ((\mathbb{Z}^+ \times \mathbb{Z}^+) \cup \{(2k, 0) \mid k \in \mathbb{N}\} \cup \{(0, 2k) \mid k \in \mathbb{N}\}, +)$ and its reachability graph:



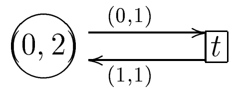

So, as it is illustrated in Example 3.1 and Lemma 3.2, cancellativity of used commutative monoids expresses an important border in the modeling power of p/t nets with generalized underlying algebra. However, according to Example 3.3, which illustrates a p/t net over a cancellative commutative monoid where the comb property does not hold, there has to exist another border in the modeling power of p/t nets with generalized algebra.

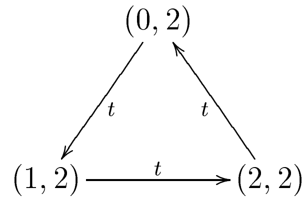
Here we have to mention an approach described in series of papers [3,2,21], where monoids being cones of groups are considered as the underlying algebra.

Very briefly, recall that a *cone group* is a system $(G, H, +, 0)$, where $(G, +, 0)$ is a group, $H \subseteq G$ and $(H, +|_{H\times H}, 0)$ is a monoid such that $\forall a, b \in H : a + b = 0 \Longrightarrow a = b = 0$. As we have already mentioned in the paper we investigate p/t nets over commutative monoids. Recall due to [6] that a commutative monoid $\mathcal{H} = (H, +)$ is a cone of an Abelian group if and only if it is cancellative and $\forall a, b \in H : a + b = 0 \Longrightarrow a = b = 0$.

Now, the question is whether p/t nets over cones of Abelian groups represent a border in the modeling power of p/t nets with generalized algebra. Unfortunately, it is easy to see that the monoid from Example 3.3 is a cone of Abelian group $\mathbb{Z} \oplus \mathbb{Z}$. However, the fact that commutative cones contain no element (except neutral element) along with its inverse could indicate that reachability graphs of p/t nets over commutative cones contain no simple cycles. As shows the following example, this is an erroneous fiction.

**Example 3.4** In the figure bellow there is a marked p/t net with one transition over commutative cone $\Gamma(p) = ((\mathbb{Z}_3 \times \mathbb{Z}^+) \cup \{(0, 0)\}, +)$, which is a cone of group $\mathbb{Z}_3 \oplus \mathbb{Z}$; together with the reachability graph of the net:




**Remark 3.5** Recall that we use operator $+$ freely to denote a binary operation. Here, saying that $\Gamma(p)$ is a cone of group $\mathbb{Z}_3 \oplus \mathbb{Z}$ we determine that operator $+$ denotes the restriction of the operation of group $\mathbb{Z}_3 \oplus \mathbb{Z}$ (that is modulo 3 addition for the first element and the standard addition for the second element) on set $(\mathbb{Z}_3 \times \mathbb{Z}^+) \cup \{(0,0)\}$, while in Example 3.3 operator $+$ denotes the restriction of the operation of group $\mathbb{Z} \oplus \mathbb{Z}$ on set $(\mathbb{Z}^+ \times \mathbb{Z}^+) \cup \{(2k,0) \mid k \in \mathbb{N})\} \cup \{(0,2k) \mid k \in \mathbb{N})\}$.

As the previous examples show, p/t nets over cones of Abelian groups enable us to model a quite wide range of systems.

Moreover, we have that given an arbitrary cancellative commutative monoid $(H, +_H, 0_H)$ the monoid $((H \times \mathbb{Z}^+) \cup \{(0_H, 0)\}, +, (0_H, 0))$, where operator $+$ denotes the restriction of the operation of monoid $(H, +_H, 0_H) \oplus \mathbb{N}$ on set $(H \times \mathbb{Z}^+) \cup \{(0_H, 0)\}$, is a commutative cone.

So, given an arbitrary p/t net $\mathcal{GN} = (\mathcal{H}, T, I, O)$ over a cancellative commutative monoid $\mathcal{H} = (H, +_H, 0_H)$, taking the p/t net $\mathcal{GN}' = (\mathcal{H}', T, I', O')$ with $\mathcal{H}' = ((H \times \mathbb{Z}^+) \cup \{(0_H, 0)\}, +, (0_H, 0))$ and $I', O'$ such that for some fixed $k \in \mathbb{Z}^+$ there holds: $\forall t \in T : I'(t) = (I(t), k) \wedge O'(t) = (O(t), k)$, we have that the reachability graph of $\mathcal{N}'_G$ consists of infinite many mutually isolated copies of the reachability graph of $\mathcal{N}_G$ and a copy of its subsystem consisting only of transitions in the form $I'(t) \xrightarrow{t} O'(t)$.

Formally, taking the reachability graph $(E, T, \longrightarrow)$ of the net $\mathcal{GN}$, we have that the reachability graph of the net $\mathcal{GN}'$ differs only in some isolated states from transition system $(S, T, \longrightarrow ')$ such that

$$S = (\cup_{i>k}(E \times \{i\}) \cup \{(I(t), k) \mid t \in T\} \cup \{(O(t), k) \mid t \in T\})$$

and

$$\longrightarrow ' = (\cup_{i>k}(\longrightarrow_i) \cup \{(I(t), k) \xrightarrow{t} (O(t), k) \mid t \in T\}),$$

where $\forall i > k$ relation $\longrightarrow_i \subseteq (E \times \{i\}) \times T \times (E \times \{i\})$ is defined as follows:

$$(M, i) \xrightarrow{t}_i (M', i) \iff M \xrightarrow{t} M'$$

Clearly, for all $i, j > k : i \neq j \implies \longrightarrow_i \cap \longrightarrow_j = \emptyset$.

We say that two pointed transition systems $(S, L, \longrightarrow, s_0)$ and $(S', L', \longrightarrow ', s'_0)$ are *isomorphic* iff there exist bijections $\alpha : S_1 \to S'_1$ and $\beta : L_1 \to L'_1$ such that $\forall (s, a, s') \in S \times L \times S : s \xrightarrow{a} s' \iff \alpha(s) \xrightarrow{\beta(a)} {}'\alpha(s'))$ and $\alpha(s_0) = s'_0$.

Now, taking an initial marking $M_0 \in H$ and any $i > k$, it is easy to check that reachability graph of marked p/t net $(\mathcal{GN}, M_0)$ is isomorphic to reachability graph of marked p/t net $(\mathcal{GN}', (M_0, i))$.

Given classes $\mathbf{X}$ and $\mathbf{Y}$ of pointed transitions systems, we write $\mathbf{X} \subseteq \mathbf{Y}$ if and only if for every pointed transition system $\mathcal{PS} \in \mathbf{X}$ there exists a pointed transition system $\mathcal{PS}' \in \mathbf{Y}$ which is isomorphic to $\mathcal{PS}$. Moreover, we write:

- $\mathbf{X} = \mathbf{Y}$ to denote that classes $\mathbf{X}$ and $\mathbf{Y}$ are *equal (up to isomorphism)*, *i.e.*, $\mathbf{X} \subseteq \mathbf{Y} \ \wedge \ \mathbf{Y} \subseteq \mathbf{X}$;
- $\mathbf{X} \subset \mathbf{Y}$ to denote that class $\mathbf{X}$ is a *proper subclass (up to isomorphism)* of class $\mathbf{Y}$, *i.e.*, $\mathbf{X} \subseteq \mathbf{Y} \ \wedge \ \mathbf{Y} \nsubseteq \mathbf{X}$.

Denoting by **CCCM** the class of reachability graphs of all marked p/t nets over cancellative commutative monoids, which are cones of Abelian groups, and by **CCM** the class of reachability graphs of all marked p/t nets over cancellative commutative monoids, the previous construction implies the following lemma.

**Lemma 3.6** **CCCM** = **CCM**.

Thus, one could say that the importance of the fact that the underlying cancellative commutative monoid, say $(H, +, 0)$, in a marked p/t net is a cone is meaningless; in other words, the property $\forall a, b \in H : a+b = 0 \Longrightarrow a = b = 0$ is redundant!

Now one could ask whether we can set an algebraic property which satisfies the comb property as well as absence of simple cycles in the reachability graphs. Before we give an answer to this question, let us recall a bit more about ordering of groups defined by cones.

In fact, given an Abelian cone group $(G, H, +)$, cone $H$ defines a partial order $\leq$ on the group $(G, +)$ such that $\forall a, b \in G : a \leq b \iff b - a \in H$ which is *monotone*, *i.e.*, $\forall a, b, c \in G : a \leq b \Longrightarrow a + c \leq b + c$. On the other hand, given a monotone partial order $\leq$ on a group $(G, +)$, set $\{a \mid 0 \leq a\}$ of all nonnegative elements creates a cone. A cone $H$ defines a full order on $(G, +)$ iff $G = H \cup -H$, where $-H = \{a \mid -a \in H\}$. Then the group as well as the cone are said to be *fully ordered*. For more details about partially ordered groups see *e.g.* [6].

Here recall, that it is easy to show torsion-freeness of every Abelian group which is a direct product of fully ordered Abelian groups.

Thus, looking at the proof of the comb property and the absence of simple cycles of standard p/t nets, which only uses full order of integers $\mathbb{Z}$ and torsion-freeness of the embedding $\mathbb{Z}^P$ of the underlying monoid $\mathbb{N}^P$, we can formulate the following claim.

**Lemma 3.7** *Reachability graph of a marked p/t net $\mathcal{MN}_G = (\mathcal{H}, T, I, O, M_0)$ such that $\mathcal{H}$ is a direct product of a collection of fully ordered commutative cones preserves the comb property and contains no simple cycles.*

## 4 Conclusions and further research

In the paper we have studied properties of Petri nets with generalized underlying algebra. Chosen a tuple of properties defined over labeled transition systems, namely determinism, commutativity, the comb property, and absence of simple cycles, we have investigated whether reachability graphs of all p/t nets over commutative monoids (shortly **CM**), cancellative commutative monoids (**CCM**), cones of Abelian groups (**CCCM**), and fully ordered cones of Abelian groups (**FOCCCM**) preserve these properties. The obtained results are summarized in the table below.

| algebra ∖ property | determinism | commutativity | comb | no simple cycles |
|---|---|---|---|---|
| **CM** | not | not | not | not |
| **CCM** | yes | yes | not | not |
| **CCCM** | yes | yes | not | not |
| **FOCCCM** | yes | yes | yes | yes |

According to these results, and after investigation of the relationships between classes **CCM** and **CCCM** we can state the following containment relationship (up to isomorphism) between reachability graphs of p/t nets over considered classes of commutative monoids.

**Theorem 4.1** **FOCCCM** $\subset$ **CCCM** $=$ **CCM** $\subset$ **CM**

Clearly, all results presented in the paper except Lemma 3.6, *i.e.*, except the equality in the previous theorem, hold also for unmarked p/t nets. The other message of the Lemma 3.6 is that although one could propose that more general algebra induces larger class of nets there are cases when it is not true.

In the paper we have discussed only algebraic structures associated with places of Petri nets. However, structure of net transitions can be also generalized. More precisely, as it is pointed out in [16], we can see the underlying algebra of standard Petri nets as an $\mathbb{N}$-semimodule $\mathbb{N}^P$ over semiring $\mathbb{N}$[8]. Replacing $\mathbb{N}$-semimodule $\mathbb{N}^P$ over semiring $\mathbb{N}$ by arbitrary semimodule over arbitrary semiring, Meseguer and Montanari get so called $R$-Petri nets. In such nets states (elements of carrier of semimodule) are changed by occurrences of steps or firing vectors, which are functions from set of transitions to the carrier of the semiring. The main purpose of steps is to deal with concurrent occurrences of transitions and/or quantities of transitions. Here let us mention that for investigation of $R$-Petri nets one should choose (an extension of) non-interleaving semantics, which enables to deal with concurrent occurrence of transitions, *e.g.* an extension of step transition systems [18], transition systems with independence [23], automata with concurrency

[8] For formal definition of semimodule over semiring see e.g. [15]

relation [5], or simply graphs equipped with an algebraic structure [16].

Finally, let us remark that also investigation of properties and containment relation for related classes of monoidal categories, which represent semantics of Petri nets dealing with both concurrent and sequential occurrences of transitions (concurrent and sequential composition of processes), could be challenging task for further research.

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